New Keynesian Model: Approximation

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NK Model is often analyzed in LQ—Linear-Quadratic—framework.

- Easier to interpret.
- Easier to solve.

Technical takeaways from this lecture:

- Learn how to take the first-order approximation to the equilibrium conditions.
- Learn how to take the second-order approximation to the household's welfare.

Substantive takeaways from this lecture (I):

- Output today depends on the expected sum of future real interest rates.
- Inflation today depends on the expected discounted sum of future marginal costs.

Substantive takeaways from this lecture (II):

- Welfare today depends on the expected discounted sum of future per-period utility where…
  - Per-period utility depends on inflation and output volatility.
  - Per-period utility also depends on the level of output if steady state is not efficient (if tau is not 1/(theta-1)).

Plan

(1) Log-linear Approximation of the Equilibrium Conditions

(2) Quadratic Approximation of the Household's Welfare

Plan

(1) Log-linear Approximation of the Equilibrium Conditions

(2) Quadratic Approximation of the Household's Welfare

Math Background

Log-linearization =

First-order Taylor expansion

+

Log-approximation of deviation

First-order Taylor expansion:

Let f(x) be a continuous and differentiable function of x. Then,

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + (\text{higher order terms})$$

Let f_x denote $\frac{\partial f}{\partial x}$.

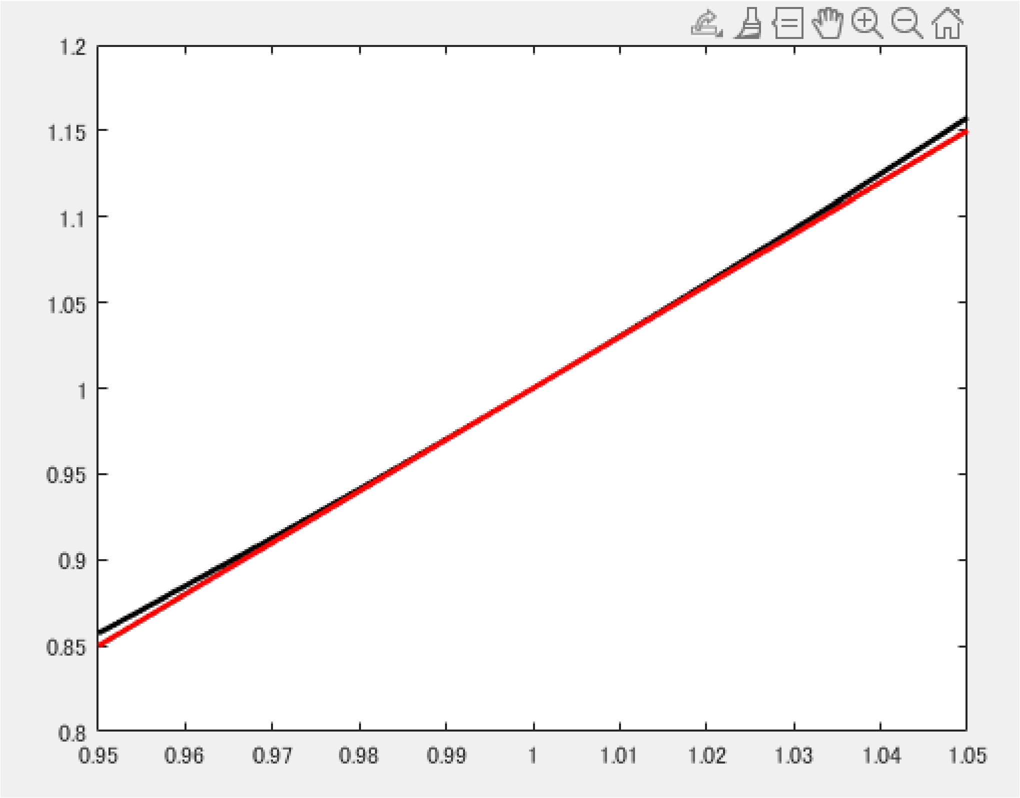
Example:

$$f(x) = x^3$$

$$f_x = 3x^2$$

Let's take the first-order Taylor expansion around $x_0 = 1$.

$$\begin{aligned} f(x) &\approx f(1) + f_x(1)(x - 1) + (\mathrm{h.o.t.}) \\ &= 1^3 + 3 * 1^2(x - 1) + (\mathrm{h.o.t.}) \\ &= 1 + 3(x - 1) + (\mathrm{h.o.t.}) \\ &= -2 + 3x + (\mathrm{h.o.t.}) \end{aligned}$$



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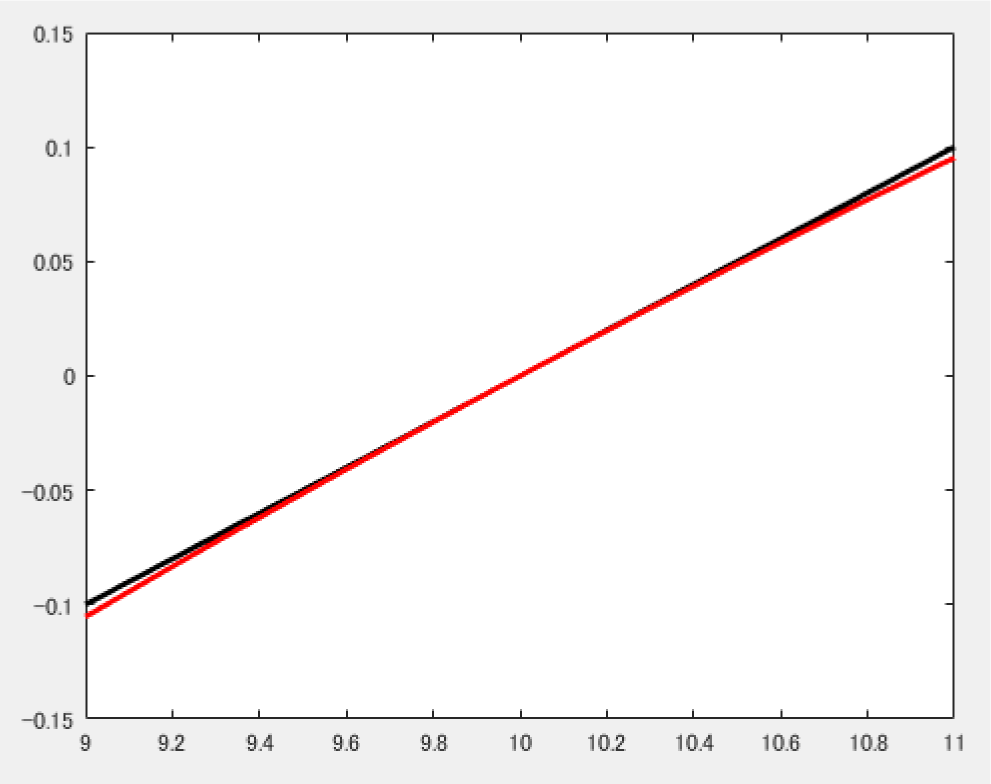
1.02
1.03
1.04
1.05

Log-approximation of percent deviation:

$$\frac{(x - x_0)}{x_0} \approx \log(x) - \log(x_0)$$

Example:

$$\frac{(x - 10)}{10} \approx \log(x) - \log(10)$$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9.2
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10
10.2
10.4
10.6
10.8
11

First-order Taylor expansion + Log-approximation of percent changes:

$$f(x) \approx f(x_0) + f_x(x_0)(x - x_0) + (h.o.t.)$$

$$= f(x_0) + f_x(x_0)x_0 \frac{(x - x_0)}{x_0} + (h.o.t.)$$

$$\approx f(x_0) + f_x(x_0)x_0(\log(x) - \log(x_0)) + (h.o.t.)$$

$$\approx f(x_0) + f_x(x_0)x_0\, \hat{x} + (h.o.t.)$$

where $\hat{x} := \log(x) - \log(x_0)$.

To summarize

Log-linearization:

$$f(x) \approx f(x_0) + f_x(x_0)x_0\,\hat{x} + (h.o.t.)$$

where $\hat{x} := \log(x) - \log(x_0)$.

Second-order Taylor expansion:

Let f(x) be a continuous and differentiable function of x. Then,

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0)(x - x_0)^2 + (\text{h.o.t.})$$

Let f_x and f_{xx} denote $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$, respectively.

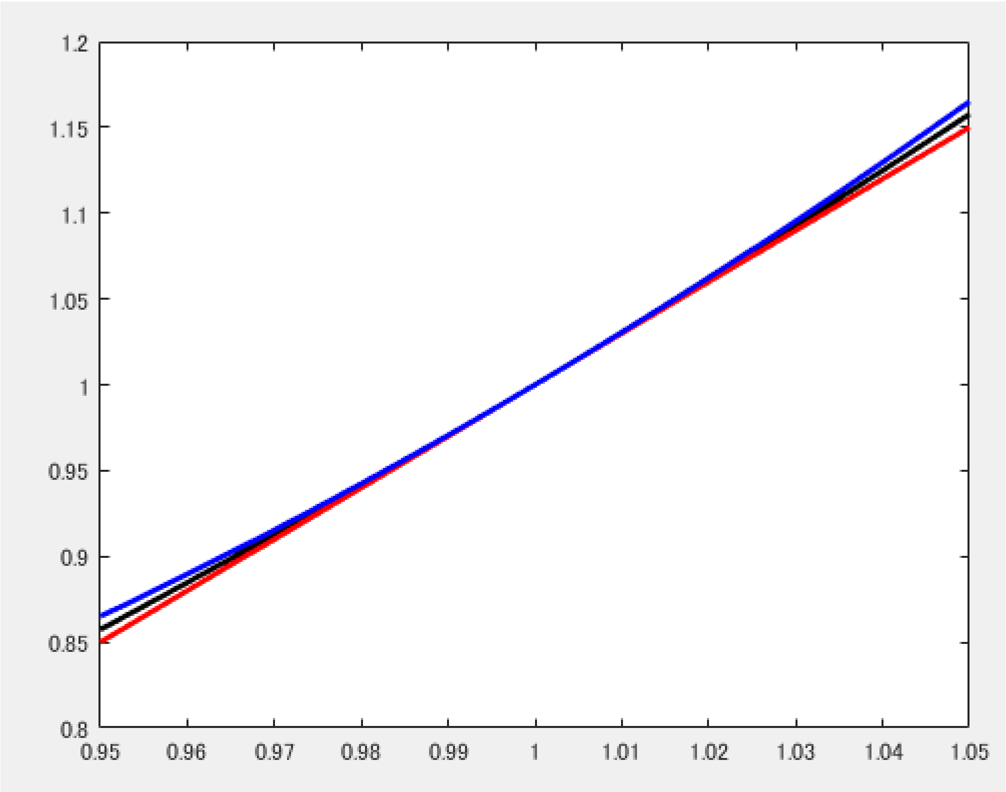
Example:

$$f(x) = x^3$$

$$f_x = 3x^2$$
$$f_{xx} = 6x$$

Let's take the second-order Taylor expansion around $x_0 = 1$.

$$f(x) \approx f(1) + f_x(1)(x-1) + f_{xx}(1)(x-1)^2 + (\mathrm{h.o.t.})$$
$$= 1^3 + 3 * 1^2(x-1) + 6 * 1^3(x-1)^2 + (\mathrm{h.o.t.})$$
$$= 1 + 3(x-1) + 6(x-1)^2 + (\mathrm{h.o.t.})$$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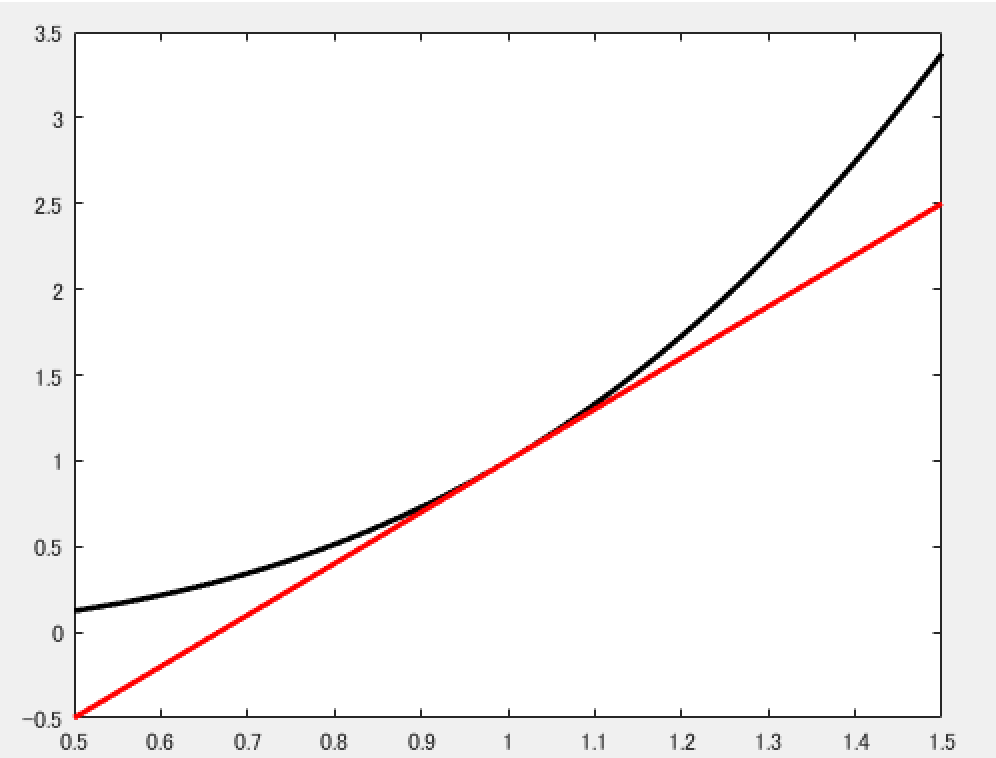
Combined with log-approximation of percent changes⋯

$$f(x) \approx f(x_0) + f_x(x_0)x_0\,\hat{x} + f_{xx}(x_0)x_0^2\,\hat{x}^2 + (h.o.t.)$$

where $\hat{x} := \log(x) - \log(x_0)$.

Word of caution on Taylor expansion.

They are good “local” approximation, but may not be good globally.


3.5
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2
1.5
1
0.5
0
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1
1.1
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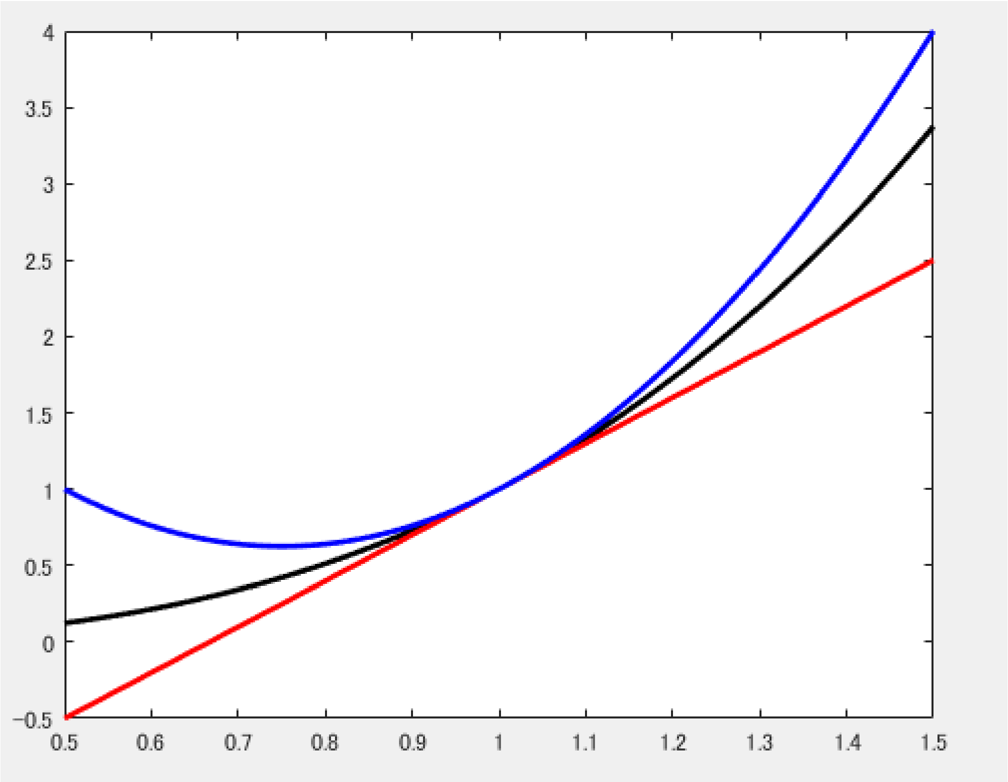
1.3
1.4
1.5



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0.5
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1
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1.2
1.3
1.4
1.5

End of Math Background

Recall:

$$C_t^{-\chi_c} = \beta\delta_t R_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c}$$

$$\frac{Y_t}{C_t^{\chi_c}}[\varphi(\Pi_t - 1)\Pi_t - (1+\tau)(1-\theta) - \theta w_t]$$

$$= \beta\delta_t \frac{Y_{t+1}}{C_{t+1}^{\chi_c}} \varphi(\Pi_{t+1} - 1)\Pi_{t+1}$$

$$Y_t = C_t + \frac{\varphi}{2}[\Pi_t - 1]^2 Y_t$$

$$Y_t = N_t$$

$$R_t = \max\left[\frac{1}{\beta}\Pi_t^{\Phi_\pi}, 1\right]$$

$$U_t = \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n}$$

Loglinearize the consumption Euler equation **around the zero-inflation steady state:**

$$C_t^{-\chi_c} R_t^{-1} = \beta \delta_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$

$$\text{LHS} = f(C_t, R_t)$$
$$\text{RHS} = g(\delta_t, C_{t+1}, \Pi_{t+1})$$

$$f_C = -\chi_c C_t^{-\chi_c - 1} R_t^{-1}$$
$$f_R = -C_t^{-\chi_c} R_t^{-2}$$

$$g_\delta = \beta C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$
$$g_C = -\chi_c \beta C_{t+1}^{-\chi_c - 1} \Pi_{t+1}^{-1}$$
$$g_\Pi = -\beta C_{t+1}^{-\chi_c} \Pi_{t+1}^{-2}$$

Notation: $h_x \coloneqq \frac{\partial h}{\partial x}$ for any functionh and for any variable x.

$$\mathrm{LHS} \approx f(C_{ss}, R_{ss}) + f_{C,ss} C_{ss} \hat{C}_t + f_{R,ss} R_{ss} \hat{R}_t$$

$$= f(C_{ss}, R_{ss}) + f(C_{ss}, R_{ss})\left[-\chi_c \hat{C}_t\right] + f(C_{ss}, R_{ss})\left[-\hat{R}_t\right]$$

$$\mathrm{RHS} \approx g(\delta_{ss}, C_{ss}, \Pi_{ss}) + g_{\delta,ss} \delta_{ss} \widehat{\delta_t} + g_{C,ss} C_{ss} \hat{C}_{t+1} + g_{\Pi,ss} \Pi_{ss} \hat{\Pi}_{t+1}$$

$$= g(\delta_{ss}, C_{ss}, \Pi_{ss}) + g(\delta_{ss}, C_{ss}, \Pi_{ss})\widehat{\delta_t} + g(\delta_{ss}, C_{ss}, \Pi_{ss})\left[-\chi_c \hat{C}_{t+1}\right] + g(\delta_{ss}, C_{ss}, \Pi_{ss})\left[-\hat{\Pi}_{t+1}\right]$$

Using $f(C_{ss}, R_{ss}) = g(\delta_{ss}, C_{ss}, \Pi_{ss})$ and rearranging terms,

$$\hat{C}_t = \hat{C}_{t+1} - \frac{1}{\chi_c}\left[\widehat{R_t} - \hat{\Pi}_{t+1} + \widehat{\delta_t}\right]$$

Loglinearize the intratemporal optimality condition **around the zero-inflation steady state:**

$$w_t = N_t^{\chi_n} C_t^{\chi_c}$$

$$\text{LHS} = f(w_t)$$
$$\text{RHS} = g(C_t, N_t)$$

$$f_w = 1$$
$$g_C = \chi_c N_t^{\chi_n} C_t^{\chi_c - 1}$$
$$g_N = \chi_n N_t^{\chi_n - 1} C_t^{\chi_c}$$

$$\text{LHS} \approx f(w_{ss}) + f_{w,ss} w_{ss} \widehat{w}_t$$
$$= f(w_{ss}) + f(w_{ss}) \widehat{w}_t$$

$$\text{RHS} \approx g(C_{ss}, N_{ss}) + g_{C,ss} C_{ss} \widehat{C}_t + g_{N,ss} N_{ss} \widehat{N}_t$$
$$= g(C_{ss}, N_{ss}) + g(C_{ss}, N_{ss}) \left[\chi_c \widehat{C}_t\right] + g(C_{ss}, N_{ss}) \left[\chi_N \widehat{N}_t\right]$$

Using $f(w_{ss}) = g(C_{ss}, N_{ss})$ and rearranging terms,

$$\widehat{w}_t = \chi_c \widehat{C}_t + \chi_n \widehat{N}_t$$

Loglinearize the optimality condition of intermediate-goods producers **around the zero-inflation steady state:**

$$Y_t C_t^{-\chi_c}[\varphi(\Pi_t - 1)\Pi_t - (1-\theta)(1+\tau) - \theta w_t]$$

$$= \beta Y_{t+1} C_{t+1}^{-\chi_c} \varphi(\Pi_{t+1} - 1)\Pi_{t+1}$$

$$\mathrm{LHS} = f(Y_t, C_t, \Pi_t, w_t)$$
$$\mathrm{RHS} = g(Y_{t+1}, C_{t+1}, \Pi_{t+1})$$

$$f_Y = C_t^{-\chi_c}[\varphi(\Pi_t - 1)\Pi_t - (1-\theta)(1+\tau) - \theta w_t]$$

$$f_C = -\chi_c Y_t C_t^{-\chi_c - 1}[\varphi(\Pi_t - 1)\Pi_t - (1-\theta)(1+\tau) - \theta w_t]$$

$$f_\Pi = Y_t C_t^{-\chi_c}\varphi[2\Pi_t - 1]$$

$$f_w = -Y_t C_t^{-\chi_c}\theta$$

$$g_Y = \beta C_{t+1}^{-\chi_c}\varphi(\Pi_{t+1} - 1)\Pi_{t+1}$$

$$g_C = -\chi_c \beta Y_{t+1} C_{t+1}^{-\chi_c - 1}\varphi(\Pi_{t+1} - 1)\Pi_{t+1}$$

$$g_\Pi = \beta Y_{t+1} C_{t+1}^{-\chi_c}\varphi[2\Pi_{t+1} - 1]$$

Note that $\Pi_{ss} = 1$ and $w_{ss} = \frac{(\theta-1)(1+\tau)}{\theta}$. Thus,

$$f_{Y,ss} = 0$$
$$f_{C,ss} = 0$$
$$f_{\Pi,ss} = Y_{ss} C_{ss}^{-\chi_c} \varphi$$
$$f_{w,ss} = -Y_{ss} C_{ss}^{-\chi_c} \theta$$

$$g_{Y,ss} = 0$$
$$g_{C,ss} = 0$$
$$g_{\Pi,ss} = \beta Y_{ss} C_{ss}^{-\chi_c} \varphi$$

$$\text{LHS} \approx f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) + f_{\Pi,ss}\Pi_{ss}\widehat{\Pi}_t + f_{w,ss}w_{ss}\widehat{w}_t$$
$$= f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) + Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\widehat{\Pi}_t - Y_{ss}C_{ss}^{-\chi_c}\theta w_{ss}\widehat{w}_t$$

$$\text{RHS} \approx g(Y_{ss}, C_{ss}, \Pi_{ss}) + g_{\Pi,ss}\Pi_{ss}\widehat{\Pi}_{t+1}$$
$$= g(Y_{ss}, C_{ss}, \Pi_{ss}) + \beta Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\widehat{\Pi}_{t+1}$$

Using $f(Y_{ss}, C_{ss}, \Pi_{ss}, w_{ss}) = g(Y_{ss}, C_{ss}, \Pi_{ss})$, $\Pi_{ss} = 1$, $w_{ss} = \frac{(\theta-1)(1+\tau)}{\theta}$ and rearranging terms,

$$Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\widehat{\Pi}_t - Y_{ss}C_{ss}^{-\chi_c}\theta w_{ss}\widehat{w_t} = Y_{ss}C_{ss}^{-\chi_c}\varphi\Pi_{ss}\widehat{\Pi}_{t+1}$$

$$\Leftrightarrow \varphi\widehat{\Pi}_t - \theta\frac{(\theta-1)(1+\tau)}{\theta}\widehat{w_t} = \beta\varphi\widehat{\Pi}_{t+1}$$

$$\Leftrightarrow \widehat{\Pi}_t = \frac{(\theta-1)(1+\tau)}{\varphi}\widehat{w_t} + \beta\widehat{\Pi}_{t+1}$$

Loglinearize the aggregate production function **around the zero-inflation steady state:**

$$Y_t = N_t$$

$$LHS = f(Y_t)$$
$$RHS = g(N_t)$$

$$f_Y = 1$$
$$g_N = 1$$

$$\begin{aligned} \text{LHS} &\approx f(Y_{ss}) + f_{Y,ss} Y_{ss} \hat{Y}_t \\ &= f(Y_{ss}) + f(Y_{ss}) \hat{Y}_t \end{aligned}$$

$$\begin{aligned} \text{RHS} &\approx g(N_{ss}) + g_{N,ss} N_{ss} \hat{N}_t \\ &= g(N_{ss}) + g(N_{ss}) \hat{N}_t \end{aligned}$$

Using $f(Y_{ss}) = g(N_{ss})$, we get

$$\hat{Y}_t = \hat{N}_t$$

Loglinearize the aggregate resource constraint **around the zero-inflation steady state:**

$$Y_t = C_t + \frac{\varphi}{2}[\Pi_t - 1]^2 Y_t$$

$$\text{LHS} = f(Y_t)$$

$$\text{RHS} = g(C_t, Y_t, \Pi_t)$$

$$f_Y = 1, \quad g_C = 1$$

$$g_Y = \frac{\varphi}{2}[\Pi_t - 1]^2$$

$$g_\Pi = \varphi[\Pi_t - 1]Y_t$$

Note that $g_{Y,ss} = g_{\Pi,ss} = 0$ because $\Pi_{ss} = 1$.

$$\begin{aligned} \text{LHS} &\approx f(Y_{ss}) + f_{Y,ss} Y_{ss} \widehat{Y}_t \\ &= f(Y_{ss}) + Y_{ss} \widehat{Y}_t \end{aligned}$$

$$\begin{aligned} \text{RHS} &\approx g(C_{ss}, Y_{ss}, \Pi_{ss}) + g_{C,ss} C_{ss} \widehat{C}_t \\ &= g(C_{ss}, Y_{ss}, \Pi_{ss}) + C_{ss} \widehat{C}_t \end{aligned}$$

Note $f(Y_{ss}) = g(C_{ss}, Y_{ss}, \Pi_{ss})$. Also, $Y_{ss} = C_{ss}$ because $\Pi_{ss} = 1$. We obtain

$$\widehat{Y}_t = \widehat{C}_t$$

Loglinearize the Taylor rule **around the zero-inflation steady state:**

$$R_t = \max\left[1, \frac{1}{\beta}\Pi_t^{\phi}\right]$$

$$\text{LHS} = f(R_t)$$
$$\text{RHS} = g(\Pi_t)$$

$$f_R = 1$$
$$g_\Pi = \phi\frac{1}{\beta}\Pi_t^{\phi-1}$$

$$\begin{aligned} \mathrm{LHS} &\approx f(R_{ss}) + f_{R,ss} R_{ss} \widehat{R}_t \\ &= f(R_{ss}) + 1 * R_{ss} \widehat{R}_t \\ &= f(R_{ss}) + f(R_{ss}) \widehat{R}_t \end{aligned}$$

$$\begin{aligned} \mathrm{RHS} &\approx g(\Pi_{ss}) + g_{\Pi,ss} \Pi_{ss} \widehat{\Pi}_t \\ &= g(\Pi_{ss}) + \phi \frac{1}{\beta} \Pi_{SS}^{\phi-1} \Pi_{ss} \widehat{\Pi}_t \\ &= g(\Pi_{ss}) + \phi \frac{1}{\beta} \Pi_{SS}^{\phi} \widehat{\Pi}_t \\ &= g(\Pi_{ss}) + g(\Pi_{ss}) \phi \widehat{\Pi}_t \end{aligned}$$

$$\widehat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi \widehat{\Pi}_t\right]$$

So far, we have

$$\hat{C}_t = \hat{C}_{t+1} - \frac{1}{\chi_c}\left[\hat{R}_t - \hat{\Pi}_{t+1} + \hat{\delta}_t\right]$$

$$\hat{w}_t = \chi_c \hat{C}_t + \chi_n \hat{N}_t$$

$$\hat{\Pi}_t = \frac{(\theta - 1)(1 + \tau)}{\varphi} \hat{w}_t + \beta \hat{\Pi}_{t+1}$$

$$\hat{Y}_t = \hat{N}_t$$

$$\hat{Y}_t = \hat{C}_t$$

$$\hat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi \hat{\Pi}_t\right]$$

Eliminating $\hat{w}_t, \hat{N}_t,$ and $\hat{C}_t$, we obtain

$$\hat{Y}_t = \hat{Y}_{t+1} - \frac{1}{\chi_c}\left[\hat{R}_t - \hat{\Pi}_{t+1} + \hat{\delta}_t\right] \text{ [“IS Curve”]}$$

$$\hat{\Pi}_t = \frac{(\theta-1)(1+\tau)}{\varphi}(\chi_c + \chi_n)\hat{Y}_t + \beta\hat{\Pi}_{t+1} \text{ [“Phillips Curve”]}$$

$$\hat{R}_t = \max\left[1 - \frac{1}{\beta}, \phi\hat{\Pi}_t\right]$$

$$\hat{R}_t + r^* = \max\left[r^* + 1 - \frac{1}{\beta}, r^* + \phi\hat{\Pi}_t\right]$$

$$i_t = \max\left[\frac{1}{\beta} - 1 + 1 - \frac{1}{\beta}, r^* + \phi\hat{\Pi}_t\right]$$

$$i_t = \max\left[0, r^* + \phi\hat{\Pi}_t\right]$$

Change of notation: Using

$$y_t \coloneqq \widehat{Y}_t, \quad \pi_t \coloneqq \widehat{\Pi}_t, \quad i_t \coloneqq \widehat{R}_t + r^*$$
$$r^* \coloneqq \frac{1}{\beta} - 1, \quad \sigma \coloneqq \frac{1}{\chi_c}, \quad \kappa \coloneqq \frac{(\theta-1)(1+\tau)}{\varphi}(\chi_c + \chi_n)$$

we obtain

$$y_t = y_{t+1} - \sigma[i_t - \pi_{t+1} - r^*]$$

$$\pi_t = \kappa y_t + \beta\pi_{t+1}$$

$$i_t = \max[0, r^* + \phi\pi_t]$$

Implications of the IS curve and the Phillips curve

1. Output today depends on the sum of future real rates.

$$
\begin{aligned}
y_t &= y_{t+1} - \sigma[i_t - \pi_{t+1} - r^*] \\
&= \left[y_{t+2} - \sigma[i_{t+1} - \pi_{t+2} - r^*]\right] - \sigma[i_t - \pi_{t+1} - r^*] \\
&= \left[\left[y_{t+3} - \sigma[i_{t+2} - \pi_{t+3} - r^*]\right] - \sigma[i_{t+1} - \pi_{t+2} - r^*]\right] - \\
&\quad \sigma[i_t - \pi_{t+1} - r^*] \\
&\dots \\
&= -\sigma \sum_{k=0}^{\infty} [i_{t+k} - \pi_{t+k+1} - r^*]
\end{aligned}
$$

…using $y_\infty = y_{ss} = 0$

2. Inflation today depends on the sum of output (which is proportional to real wage---the marginal cost).

$$
\begin{aligned}
\pi_t &= \kappa y_t + \beta \pi_{t+1} \\
&= \kappa y_t + \beta[\kappa y_{t+1} + \beta \pi_{t+2}] \\
&= \kappa y_t + \beta \kappa y_{t+1} + \beta^2[\kappa y_{t+2} + \beta \pi_{t+3}] \\
&= \kappa y_t + \beta \kappa y_{t+1} + \beta^2 \kappa y_{t+2} + \cdots \\
&= \cdots \\
&= \kappa \sum_{k=0}^{\infty} \beta^k y_{t+k}
\end{aligned}
$$

$\cdots$using $\pi_\infty = \pi_{ss} = 0$

Plan

(1) Log-linear Approximation of the Equilibrium Conditions

(2) Quadratic Approximation of the Household's Welfare

Take the second-order expansion of the household utility **around the zero-inflation steady state.**

$$U(C_t, N_t) \approx U_{ss} + U_{C,ss} C_{ss} \hat{C}_t + U_{N,ss} N_{ss} \hat{N}_t$$

$$+ \frac{1}{2} U_{CC,ss} C_{ss}^2 \hat{C}_t^2 + \frac{1}{2} U_{CN,ss} C_{ss} N_{ss} \hat{C}_t \hat{N}_t$$

$$+ \frac{1}{2} U_{NC,ss} N_{ss} C_{ss} \hat{N}_t \hat{C}_t + \frac{1}{2} U_{NN,ss} N_{ss}^2 \hat{N}_t^2$$

Useful stuff (I):

$$U_{C,ss} = C_t^{-\chi_c}, \quad U_{N,ss} = -N_t^{\chi_n}, \quad U_{CC,ss} = -\chi_c C_t^{-\chi_c - 1},$$

$$U_{CN,ss} = 0, \quad U_{NC,ss} = 0, \quad U_{NN,ss} = -\chi_n N_t^{\chi_n - 1}$$

$$\frac{U_{CC,ss}}{U_{C,ss}} = \frac{-\chi_c C_{ss}^{-\chi_c - 1}}{C_{ss}^{-\chi_c}} = -\chi_c C_{ss}^{-1}$$

$$\frac{U_{NN,ss}}{U_{N,ss}} = \frac{-\chi_n N_t^{\chi_n - 1}}{-N_t^{\chi_n}} = \chi_n N_{ss}^{-1}$$

Useful stuff (II): Second-order approximation of the aggregate resource constraint.

$$N_t = C_t + \frac{\varphi}{2}(\Pi_t - 1)^2 N_t$$

$$LHS = f(N_t)$$
$$RHS = g(C_t, N_t, \Pi_t)$$

$$f_N = 1, \quad f_{NN} = 0$$
$$g_C = 1, \quad g_N = \frac{\varphi}{2}(\Pi_t - 1)^2, \quad g_\Pi = \varphi(\Pi_t - 1)N_t$$
$$g_{CC} = 0, \quad g_{CN} = 0, \quad g_{C\Pi} = 0$$
$$g_{NC} = 0, \quad g_{NN} = 0, \quad g_{N\Pi} = \varphi(\Pi_t - 1)$$
$$g_{\Pi C} = 0, \quad g_{\Pi N} = \varphi(\Pi_t - 1), \quad g_{\Pi\Pi} = \varphi N_t$$

For g, at the steady state, the only non-zero coefficient is $g_C = 1$ and $g_{\Pi\Pi} = \varphi N_{ss}$ (because $\Pi_{ss} = 1$).

$$\text{LHS} \approx f_{ss} + f_{N,ss} N_{ss} \widehat{N_t} + \frac{f_{N,ss}}{2} N_{ss}^2 \widehat{N}_t^2$$

$$= f_{ss} + N_{ss} \widehat{N_t}$$

$$\text{RHS} \approx g_{ss} + g_{C,ss} C_{ss} \widehat{C}_t + g_{N,ss} N_{ss} \widehat{N_t} + g_{\Pi,ss} \Pi_{ss} \widehat{\Pi}_t$$

$$+ \frac{1}{2} [g_{CC,ss} C_{ss}^2 \widehat{C}_t^2 + g_{CN,ss} C_{ss} N_{ss} \widehat{C}_t \widehat{N_t} + g_{C\Pi,ss} \Pi_{ss}^2 \widehat{\Pi}_t^2$$

$$+ g_{NC,ss} N_{ss} C_{ss} \widehat{N_t} \widehat{C}_t + g_{NN,ss} N_{ss}^2 \widehat{N}_t^2 + g_{N\Pi,ss} N_{ss} \Pi_{ss} \widehat{N}_t \widehat{\Pi_t}$$

$$+ g_{\Pi C,ss} \Pi_{ss} C_{ss} \widehat{\Pi_t} \widehat{C}_t + g_{\Pi N,ss} \Pi_{ss} N_{ss} \widehat{\Pi_t} \widehat{N_t}$$

$$+ g_{\Pi\Pi,ss} \Pi_{ss}^2 \widehat{\Pi}_t^2]$$

$$= g_{ss} + g_{C,ss} C_{ss} \widehat{C}_t + \frac{1}{2} g_{\Pi\Pi,ss} \Pi_{ss}^2 \widehat{\Pi}_t^2$$

$$= g_{ss} + C_{ss} \widehat{C}_t + \frac{\varphi}{2} N_{ss} \widehat{\Pi}_t^2$$

$$\Rightarrow \widehat{N_t} = \widehat{C}_t + \frac{\varphi}{2} \widehat{\Pi}_t^2$$

$$U_t - U_{ss} \approx U_{C,ss} C_{ss} \widehat{C_t} + U_{N,ss} N_{ss} \widehat{N_t} + \frac{1}{2} U_{CC,ss} C_{ss}^2 \hat{C}_t^2 + \frac{1}{2} U_{NN,ss} N_{ss}^2 \hat{N}_t^2$$

$$= U_{C,ss} C_{ss} \left(\widehat{C_t} + \frac{1}{2} \frac{U_{CC,ss} C_{ss}^2}{U_{C,ss} C_{ss}} \hat{C}_t^2 \right) + U_{N,ss} N_{ss} \left(\widehat{N_t} + \frac{1}{2} \frac{U_{NN,ss} N_{ss}^2}{U_{N,ss} N_{ss}} \hat{N}_t^2 \right)$$

$$= U_{C,ss} C_{ss} \left(\widehat{C_t} - \frac{1}{2} \chi_c \hat{C}_t^2 \right) + U_{N,ss} N_{ss} \left(\widehat{N_t} + \frac{1}{2} \chi_n \hat{N}_t^2 \right)$$

$$\frac{U_t - U_{ss}}{U_{C,ss} C_{ss}} \approx \left(\widehat{C_t} - \frac{\chi_c}{2} \widehat{C}_t^2\right) + \frac{U_{N,ss} N_{ss}}{U_{C,ss} C_{ss}} \left(\widehat{N_t} + \frac{\chi_n}{2} \widehat{N}_t^2\right)$$

$$= \left(\widehat{C_t} - \frac{\chi_c}{2} \widehat{C}_t^2\right) - w_{ss} \left(\widehat{N_t} + \frac{\chi_n}{2} \widehat{N}_t^2\right)$$

$$= \left(\widehat{C_t} - \frac{\chi_c}{2} \widehat{C}_t^2\right) - \frac{(\theta - 1)(1 + \tau)}{\theta} \left(\widehat{N_t} + \frac{\chi_n}{2} \widehat{N}_t^2\right)$$

$$= \left(\widehat{C_t} - \frac{\chi_c}{2} \widehat{C}_t^2\right)$$

$$- \frac{(\theta - 1)(1 + \tau)}{\theta} \left(\widehat{C_t} + \frac{\varphi}{2} \widehat{\Pi}_t^2 + \frac{\chi_n}{2} \left[\widehat{C_t} + \frac{\varphi}{2} \widehat{\Pi}_t^2\right]^2\right)$$

$$\approx \left(\widehat{C}_t - \frac{\chi_c}{2}\hat{C}_t^2\right) - \frac{(\theta-1)(1+\tau)}{\theta}\left(\widehat{C}_t + \frac{\varphi}{2}\widehat{\Pi}_t^2 + \frac{\chi_n}{2}\hat{C}_t^2\right)$$

$$= \frac{\theta-(\theta-1)(1+\tau)}{\theta}\widehat{C}_t$$

$$- \left(\frac{\chi_c}{2} + \frac{(\theta-1)(1+\tau)}{\theta}\frac{\chi_n}{2}\right)\hat{C}_t^2 - \frac{(\theta-1)(1+\tau)}{\theta}\frac{\varphi}{2}\widehat{\Pi}_t^2$$

$$\propto \frac{\theta-(\theta-1)(1+\tau)}{\varphi(\theta-1)(1+\tau)}\widehat{C}_t - \frac{1}{2}\left[\widehat{\Pi}_t^2 + \frac{\theta\chi_c+(\theta-1)(1+\tau)\chi_n}{\varphi(\theta-1)(1+\tau)}\hat{C}_t^2\right]$$

$$= -\frac{1}{2}\left[\widehat{\Pi}_t^2 + \frac{\chi_c+\chi_n}{\varphi}\hat{C}_t^2\right] \quad \left(\text{if } \tau = \frac{1}{\theta-1}\right)$$

Change of notation: Using

$$y_t \coloneqq \widehat{Y}_t, \quad \pi_t \coloneqq \widehat{\Pi}_t, \quad i_t \coloneqq \widehat{R}_t + r^*$$
$$r^* \coloneqq \frac{1}{\beta} - 1, \quad \sigma \coloneqq \frac{1}{\chi_c}, \quad \kappa \coloneqq \frac{\theta-1}{\varphi}(\chi_c + \chi_n)$$

we obtain

$$\frac{U_t - U_{ss}}{U_{C,ss}C_{ss}}$$
$$\propto \frac{\theta-(\theta-1)(1+\tau)}{\varphi(\theta-1)(1+\tau)} y_t - \frac{1}{2}\left[\pi_t^2 + \frac{\theta\chi_c + (\theta-1)(1+\tau)\chi_n}{\varphi(\theta-1)(1+\tau)} y_t^2\right]$$
$$= -\frac{1}{2}\left[\pi_t^2 + \frac{\chi_c + \chi_n}{\varphi} y_t^2\right] \quad \left(\text{if } \tau = \frac{1}{\theta-1}\right)$$

$$u(\pi_t, y_t) \coloneqq \lambda_y y_t - \frac{1}{2}[\pi_t^2 + \lambda y_t^2]$$

where

$$\lambda_y \coloneqq \frac{\theta - (\theta - 1)(1 + \tau)}{\varphi(\theta - 1)(1 + \tau)}$$

$$\lambda := \frac{\theta\chi_c + (\theta - 1)(1 + \tau)\chi_n}{\varphi(\theta - 1)(1 + \tau)}$$

Note that, if $\tau = \frac{1}{\theta - 1}$,

$$\lambda_y \coloneqq 0$$

$$\lambda := \frac{\chi_c + \chi_n}{\varphi}$$

$$u(\pi_t, y_t) \coloneqq \lambda_y y_t - \frac{1}{2}[\pi_t^2 + \lambda y_t^2]$$

$$= -\frac{1}{2}\left[\pi_t^2 + \lambda y_t^2 - 2\lambda_y y_t\right]$$

$$= -\frac{1}{2}\left[\pi_t^2 + \lambda y_t^2 - 2\lambda_y y_t + \lambda\left(\frac{\lambda_y}{\lambda}\right)^2 - \lambda\left(\frac{\lambda_y}{\lambda}\right)^2\right]$$

$$= -\frac{1}{2}\left[\pi_t^2 + \lambda y_t^2 - 2\lambda\frac{\lambda_y}{\lambda} y_t + \lambda\left(\frac{\lambda_y}{\lambda}\right)^2 - \lambda\left(\frac{\lambda_y}{\lambda}\right)^2\right]$$

$$= -\frac{1}{2}\left[\pi_t^2 + \lambda\left(y_t - \frac{\lambda_y}{\lambda}\right)^2 - \lambda\left(\frac{\lambda_y}{\lambda}\right)^2\right]$$

$$= -\frac{1}{2}\left[\pi_t^2 + \lambda\left(y_t - \frac{\lambda_y}{\lambda}\right)^2\right] + \frac{\lambda}{2}\left(\frac{\lambda_y}{\lambda}\right)^2$$

$$= -\frac{1}{2}[\pi_t^2 + \lambda(y_t - y^*)^2] + (\mathbf{t.i.p.})$$

t.i.p. stands for “**t**erms **i**ndependent of **p**olicy.”

With some abuse of notation, I will use “u” to denote the part of the utility flow that is independent of policy going forward. That is,

$$u(\pi_t, y_t) \coloneqq -\frac{1}{2}[\pi_t^2 + \lambda(y_t - y^*)^2]$$

where

$$y^* \coloneqq \frac{\lambda_y}{\lambda}$$

If $\tau < \frac{1}{\theta-1}$, we can show that

$$y^* > 0$$

- Thus, if $y_t = 0$, the output is too low relative to the first best.

If $\tau > \frac{1}{\theta-1}$, we can show that

$$y^* < 0$$

- Thus, if $y_t = 0$, the output is too high relative to the first best.

These implications are consistent with the steady-state analysis in the nonlinear model.

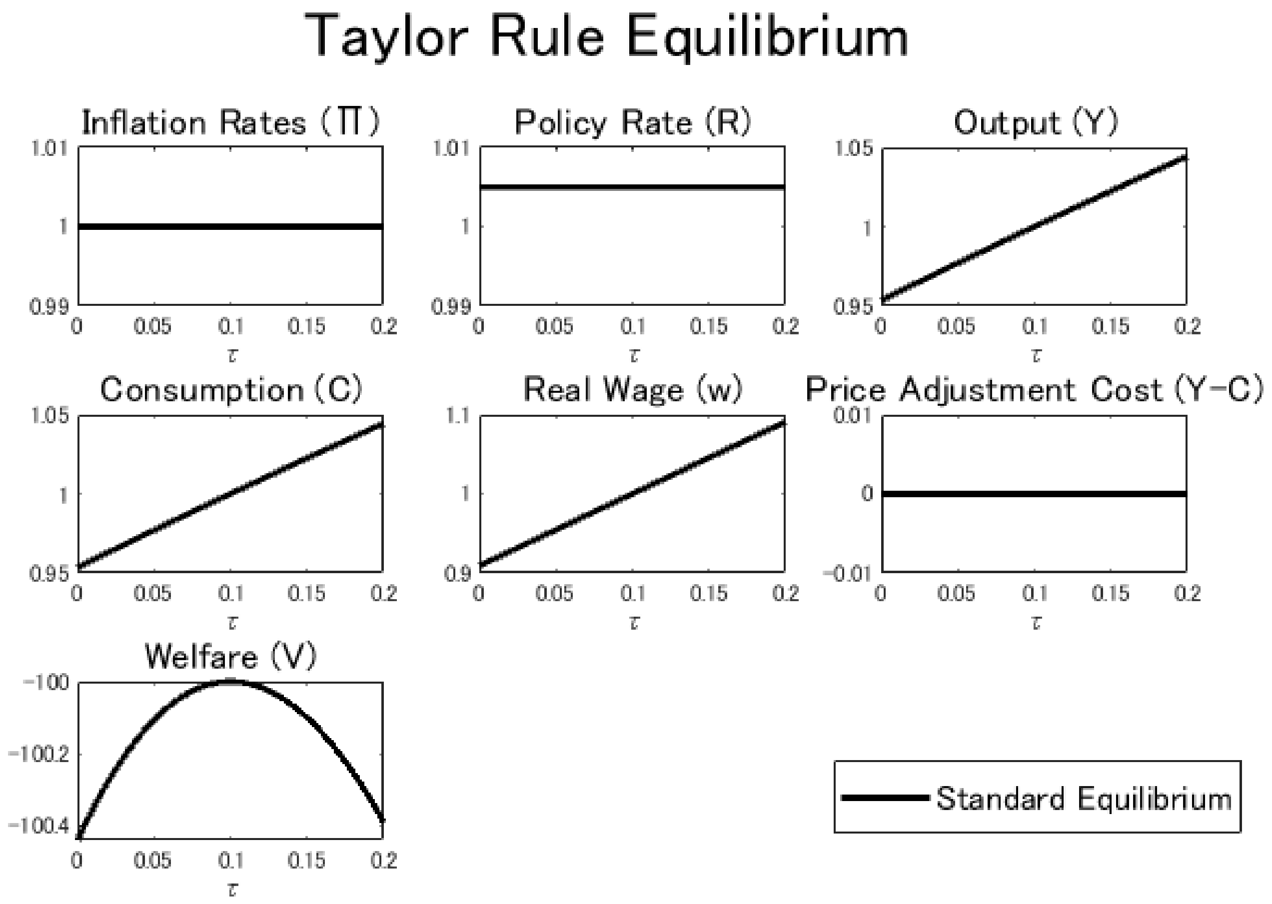

Fig. Standard Steady States with respect to τ,
where $\{\beta, \theta, \chi_c, \chi_n\} = \{0.995,11,1,1\}$

*From Slides_NK_SteadyStates.